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Three-space problem for some classes of L^1-preduals

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ABSTRACT

Following the well-known classification scheme of function spaces whose duals are isometric to $L^1(\mu)$, due to Lindenstrauss, Wulbert and Olsen [J. Lindenstrauss, D.E. Wulbert, On the classification of the Banach spaces whose duals are L_1 spaces, J. Funct. Anal. 4 (1969) 332–349; G.H. Olsen, On the classification of complex Lindenstrauss spaces, Math. Scand. 35 (1974) 237–258], in this paper we study the three-space problem for them. We investigate conditions so that a Banach space E is in a specific class if for some M-ideal $M \subset E$, both M and $E|M$ are in that class of function spaces from the classification scheme.

1. Introduction

Let E be a Banach space and let $M \subset E$ be a closed subspace. The 3-space problem in Banach space theory investigates properties (both isometric and isomorphic) that pass to E, under the assumption that both M and $E|M$ have this property. The monograph by Castillo and González [1] contains a well documented account of several such properties.

Let E be a complex Banach space such that E^* is isometric to $L^1(\mu)$ for some positive measure μ or equivalently an abstract L-space in the sense of Kakutani (see [5]). Such spaces are called L^1-preduals or Lindenstrauss spaces. Study of their structure and classification attracted a lot of attention during the 70'. Lindenstrauss and Wulbert [6] gave a classification scheme for characterizing several known classes of function spaces among the preduals of L^1. These results were extended to complex Banach spaces by Olsen [7]. See the monograph [5] for more details. The classes of L^1-preduals that we will be considering here include, the $C(X)$ spaces, the G-spaces due to Grothendieck, the C_σ, C_Σ spaces (to be defined later) and the space $A(K)$ of affine continuous functions on a compact convex set K that is also a Choquet simplex.

For a Banach space E by E_1, $S(E)$, $\partial_e E_1$ we denote the closed unit ball, the unit sphere and the set of extreme points of the unit ball respectively. We recall from [4, Chapter 1], that a closed subspace $M \subset E$ is said to be an M-ideal, if there exists a linear projection $P : E^* \to E^*$ such that $\ker P = M^\perp$ and $\|e^*\| = \|P(e^*)\| + \|e^* - P(e^*)\|$ for all $e^* \in E^*$. In this case range of P is isometric to M^* so that $E^* = M^* \oplus_1 M^\perp$ (ℓ^1-direct sum). Also, $P(e^*)$ is the unique norm preserving extension of $e^*|M$. P is a unique projection with this property and we use this notation throughout the paper. If E is an L^1-predual then for any $e^* \in \partial_e E_1^*$, $\ker(e^*)$ is an M-ideal. More generally for any weak*-closed face $F \subset E_1^*$, $M = \{e \in E\colon\ e(F) = 0\}$ is an M-ideal (see [3]). It is well known that being an L-space and hence being an L^1-predual is preserved by ranges of projections of norm one [7]. Thus if E is an L^1-predual and $M \subset E$ is an M-ideal, then as M^* and $M^\perp$ being ranges of projections of norm one in E^*, are abstract L-spaces, we get that both M, $E|M$ are L^1-preduals. Conversely if for an M-ideal, M of a Banach space E, if both M, $E|M$ are L^1-preduals, then as M^* and $M^\perp$ are abstract L-spaces and $E^* = M^* \oplus_1 M^\perp$, we have that E^* is an L-space and hence E is an L^1-predual. Thus it is interesting to ask if M and $E|M$ are in the same class of

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L^1-preduals from the classification scheme, does E belong to the same class? This turns out to be a hard problem to settle without some further assumptions.

We note that being an L^1-predual is not preserved by closed subspaces or quotients. Also by taking $E = \ell^\infty \oplus_1 \ell^\infty$, we see that a subspace and a quotient by it can be L^1-predual without the space being an L^1-predual. Thus it is more appropriate to consider the three-space problems when the subspace is an M-ideal.

We note that for the sequence spaces $c_0 \subset c$, we have that c_0 is an M-ideal but not a $C(K)$ space since its unit ball has no extreme points. However we note that the G-spaces are closed under M-ideals and quotients by M-ideals.

We show that if M, $E|M$ are $C(X)$ or $A(K)$ spaces, then so is E. If M is a C_Σ space and $E|M$ is a G space, then E is a G-space.

We also consider the 3-space problem for general Banach spaces, when the subspace is an M-ideal. We show that for a Banach space X not containing an isomorphic copy of ℓ^∞, if X and $X|M$ are isomorphic to a dual space, then so is X.

2. Main result

We now formally define various function spaces involved, see [7].

For a compact set X, $C(X)$ denotes the space of complex-valued continuous functions and for a compact convex set K that is a Choquet simplex, $A(K)$ denotes the space of complex-valued affine continuous functions on K, both the spaces equipped with the supremum norm.

Let T denote the unit circle. X is said to be a T_σ-space, if there is a continuous map $\sigma : T \times X \to X$ such that $\sigma(1, x) = x$ and $\sigma(\alpha, \sigma(\beta, x)) = \sigma(\alpha\beta, x)$, for $x \in X$ and $\alpha, \beta \in T$. $f \in C(X)$ is said to be σ-homogeneous if $f(\sigma(\alpha, x)) = \alpha f(x)$ for all $x \in X$ and $\alpha \in T$. $C_\sigma(X)$ denotes the subspace of σ-homogeneous functions. If for all $\alpha \neq 1$, $\sigma(\alpha, \cdot)$ has no fixed point in X, then this space is called a C_Σ-space.

A complex G-space is a subspace $V \subset C(X)$, satisfying a family $\mathcal{A}$ of relations, $f(x_a) = \lambda_a \alpha_a f(y_a)$ for $x_a, y_a \in X$, $\lambda_a \in [0, 1]$, $\alpha_a \in T$, $a \in \mathcal{A}$.

It is easy to see that any C_σ-space is a G-space.

Let $M \subset E$ be an M-ideal. It is easy to see that $\partial_e E_1^* = \partial_e M_1^* \cup (\partial_e M_1^\perp)$. Thus it is natural to use the characterizations of subclasses of L^1-preduals in terms of extreme points. These we now recall from [6] and [7].

We note that all the function spaces considered above are L^1-preduals. Let E be an L^1-predual space.

(1) E is isometric to a $C(X)$ space for a compact set X if and only if $\partial_e E_1 \neq \emptyset$ and $\partial_e E_1^*$ is weak*-closed.
(2) E is isometric to a C_σ space if and only if $\partial_e E_1^* \cup \{0\}$ is weak*-closed.
(3) E is isometric to a C_Σ space if and only if $\partial_e E_1^*$ is weak*-closed.
(4) E is isometric to an $A(K)$ space for a compact Choquet simplex K if and only if $\partial_e E_1 \neq \emptyset$.
(5) E is isometric to a G-space if and only if for any net $\{e_\alpha^*\} \subset \partial_e E_1^*$, $e_\alpha^* \to e^*$ in the weak*-topology implies that $e^* = 0$ or $\frac{e^*}{\|e^*\|} \in \partial_e E_1^*$.

We recall that $e \in E_1$ is a strong extreme point if $e_k \in E$, $\|e \pm e_k\| \to 1 \Longrightarrow e_k \to 0$. It is easy to see that $1 \in C(X)$ or $A(K)$ is a strong extreme point of the unit ball.

Theorem 1. *Let E be a Banach space and let $M \subset E$ be an M-ideal. If M, $E|M$ are $C(X)$ or $A(K)$ spaces, then so is E. If M is a C_Σ space and $E|M$ is a G-space, then E is a G-space.*

Proof. As already noted, the hypothesis implies that E is an L^1-predual.

Suppose M, $E|M$ are $C(X)$ spaces. Since M is an M-ideal with a strong extreme point in the unit ball, it follows from Proposition II.4.2 and Theorem II.4.4 in [4] that it is not a proper M-ideal, i.e., $E = M \oplus_\infty N$ (ℓ^∞ direct-sum) for some closed subspace N. Also as N is a $C(X)$ space, the conclusion follows.

In the case of $A(K)$ spaces, again as M is an M-ideal with a strong extreme point in the unit ball, we have $E = M \oplus_\infty N$ and since by hypothesis, N_1 also has an extreme point we get that $\partial_e E_1 \neq \emptyset$. Thus E is an $A(K)$ space for a compact Choquet simplex K.

Next suppose M is a C_Σ space and $E|M$ is a G-space. Let $\{e_\alpha^*\} \subset \partial_e E_1^*$ be a net such that $e_\alpha^* \to e^* \neq 0$ in the weak*-topology.

Suppose $e^* \notin M^\perp$. Then by weak* convergence, we get a β such that $\alpha \geqslant \beta$ implies $e_\beta^* \notin M^\perp$. As these are extreme points, we have that $\{e_\beta^*\}_{\beta \geqslant \alpha} \subset \partial_e M_1^*$. Also this subnet converges to $e^* \neq 0$ in the weak*-topology of M^*. As M is a C_Σ space, we get that $e^* \in \partial_e M_1^* \subset \partial_e E_1^*$.

Now suppose $e^* \in M^\perp$ but not in the weak*-closure of the set of extreme point of the unit ball. Again there exists an α such that $e_\beta^* \notin \partial_e M_1^\perp$ for $\beta \geqslant \alpha$. Also as these are extreme points, $\{e_\beta^*\}_{\beta \geqslant \alpha} \subset \partial_e M_1^*$. This time the subnet converges to 0 in the weak*-topology of M^*, contradicting the assumption that M is a C_Σ space. Therefore $e^* \in (\partial_e M_1^\perp)^- \subset [0, 1]\partial_e M_1^\perp \subset [0, 1]\partial_e E_1^*$.

Thus E is a G-space. $\square$

Remark 2. Arguments similar to the one above also show that if $E|M$ is a C_σ space, then so is E. For an M-ideal $M \subset E$, since $\partial_e(M^\perp)_1 \subset \partial_e E_1^*$, and $M^\perp$ is a weak*-closed set, it is easy to see that if E is any of C_σ, C_Σ, G spaces then so is $E|M$. If E is an $A(K)$ space, then it is known that $M = \{a \in A(K)\colon\ a(F) = 0\}$ for a closed face $F \subset K$. It is easy to see that $E|M$ is isometric to $A(F)$ and as any closed face of a Choquet simplex is a Choquet simplex, we have that $E|M$ is also in this class. As already noted in the introduction M need not be a $C(X)$ or an $A(K)$ space even if E is.

We do not know if the above theorem is valid for G-spaces. Our argument crucially depended on the fact that a non-zero weak*-limit is an extreme point.

We next note that G-spaces are closed under M-ideals by using the function space definition of a G-space and the description of M-ideals.

Proposition 3. *Let E be a G-space, then for any M-ideal, $M \subset E$, M is a G-space.*

Proof. We write the proof for the real scalar field. By the function space characterization of a G-space, there is a compact set X and a collection of points $\{s_a, t_a\}_{a\in\mathcal{A}} \subset X$, $\{\lambda_a\}_{a\in\mathcal{A}} \subset [-1, 1]$, such that $E = \{f \in C(X)\colon\ f(s_a) = \lambda_a f(t_a)$ for all $a \in \mathcal{A}\}$.

Since M is an M-ideal, by Proposition II.5.2 in [4], there is a closed set $D \subset X$ such that $M = \{x \in E\colon\ x(D) = 0\}$. Thus M has a function space description similar to E with the index set enlarged by takin points of D and $\lambda = 0$. Thus M is a G-space. □

Another class of function spaces for which we could not decide the three-space question is the space $C_0(X)$ of continuous functions vanishing at infinity on a locally compact space X. These are characterized (see [7, Section 5]) as those C_σ-spaces for which there is a maximal face F of the dual unit sphere such that the convex hull $CO(F \cup \{0\})$ is weak*-closed. We recall that a C_Σ-space that is a C_0-space is a $C(Y)$ for a compact set Y.

Question 4. Let E be such that for an M-ideal M, M is a C_0-space and $E|M$ is a $C(Y)$ space for a compact set Y. Is E a C_0-space?

It follows from our results here that in the above situation, E is a C_σ-space. Also as $E^* = M^* \oplus_1 M^\perp$, the assumptions imply that there is a weak*-closed face F that is maximal with respect to $M^\perp$ and a face G that is maximal with respect to M^* such that $CO(G \cup \{0\})$ is weak*-closed in M^*. We do not know how to show that $CO(F \cup G \cup \{0\})$ is a face and is weak*-closed.

The ideas from the preceding results can also be used to settle some 3-space questions in general Banach spaces, when the subspace is an M-ideal.

We recall that a Banach space X is one complemented in a dual space if it is isometric to the range of a projection of norm one, in some dual space. It is easy to see that X is one complemented in a dual space if and only if it is the range of a projection of norm one (under canonical embedding) in X^{**}.

Proposition 5. *Let X be a Banach space and $M \subset X$ is an M-ideal such that both M and $X|M$ are one complemented in some dual space. Then X is also one complemented in a dual space. In particular, if M and $X|M$ are isometric to a dual space then so is X.*

Proof. Suppose M is one complemented in L^* and $X|M$ is one complemented in N^* for some Banach spaces L, N. It follows from [4, Corollary I.1.3] that M is an M-summand in X, i.e., $X = M \oplus_\infty M'$ for some closed subspace M'. Now it is easy to see that X is one complemented in $(L \oplus_1 N)^*$. □

Remark 6. Let A be a C^*-algebra. It is well known that M-ideals in A are precisely closed two sided ideals. We recall that a von Neumann algebra is a C^*-algebra that is isometric to a dual space. Thus as a corollary to the above, we have a geometric proof of a well-known fact from C^*-algebra theory.

Let A be a C^*-algebra and $I \subset A$ be a closed two sided ideal. If I and $A|I$ are von Neumann algebras then so is A.

Conditions under which an M-ideal is an M-summand can also be used to settle some isomorphic questions. It is known that (see [1, 3.7b, p. 111]) being isomorphic to a dual space is not a 3-space property.

Proposition 7. *Suppose X has no isomorphic copy of ℓ^∞. Suppose M is an M-ideal and both M and $X|M$ are isomorphic to dual spaces. Then X is isomorphic to a dual space.*

Proof. As before we show that the hypothesis implies that M is an M-summand. If not then M is a proper M-ideal. Thus by [4, Theorem II.4.7] we have that M contains an isomorphic copy of c_0. As M is isomorphic to a dual space by a well-known result of Bessaga and Pełczyński (see [2, Corollary 6, p. 23]) we have that M and hence X has an isomorphic copy of ℓ^∞. A contradiction. □

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